Software Verification (Autumn 2015)
Lecture 16: Separation Logic
Part 2 of 2

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In the previous lecture we saw that:

- separation logic is an extension of Hoare logic for shared mutable data structures
- program states are now modelled by variable stores and heaps
- spatial connectives allow assertions to focus on resources used by programs
- frame rule enables local reasoning

Next on the agenda

(1) model of program states for separation logic ✓

(2) assertions and spatial connectives ✓

(3) axioms and inference rules ✓

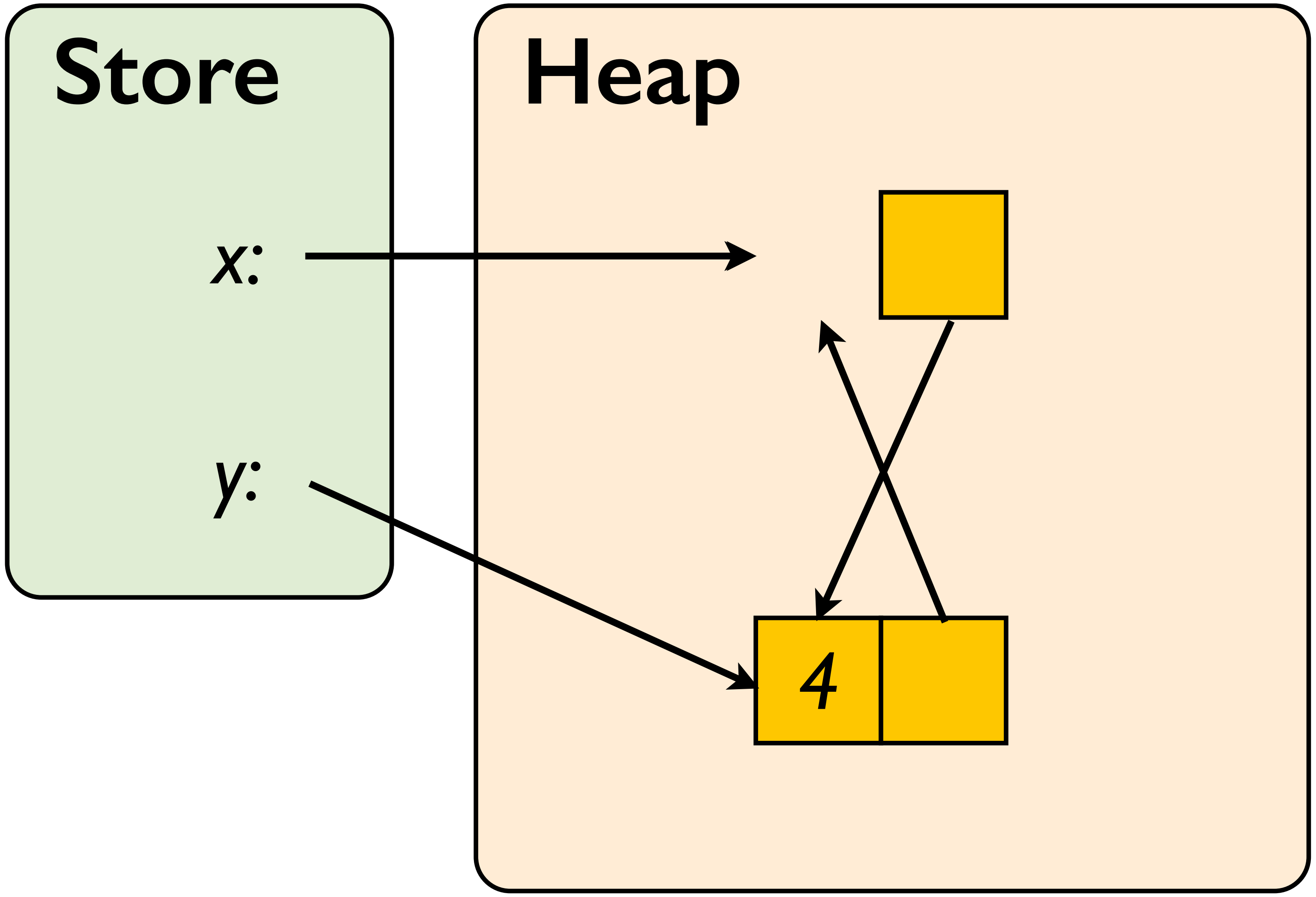
(4) program proofs

Exercise: prove this!

{emp}

x := cons(3,3);
y := cons(4,4);
[x+1] := y;
[y+1] := x;
y := x+1;
dispose x;
y := [y];

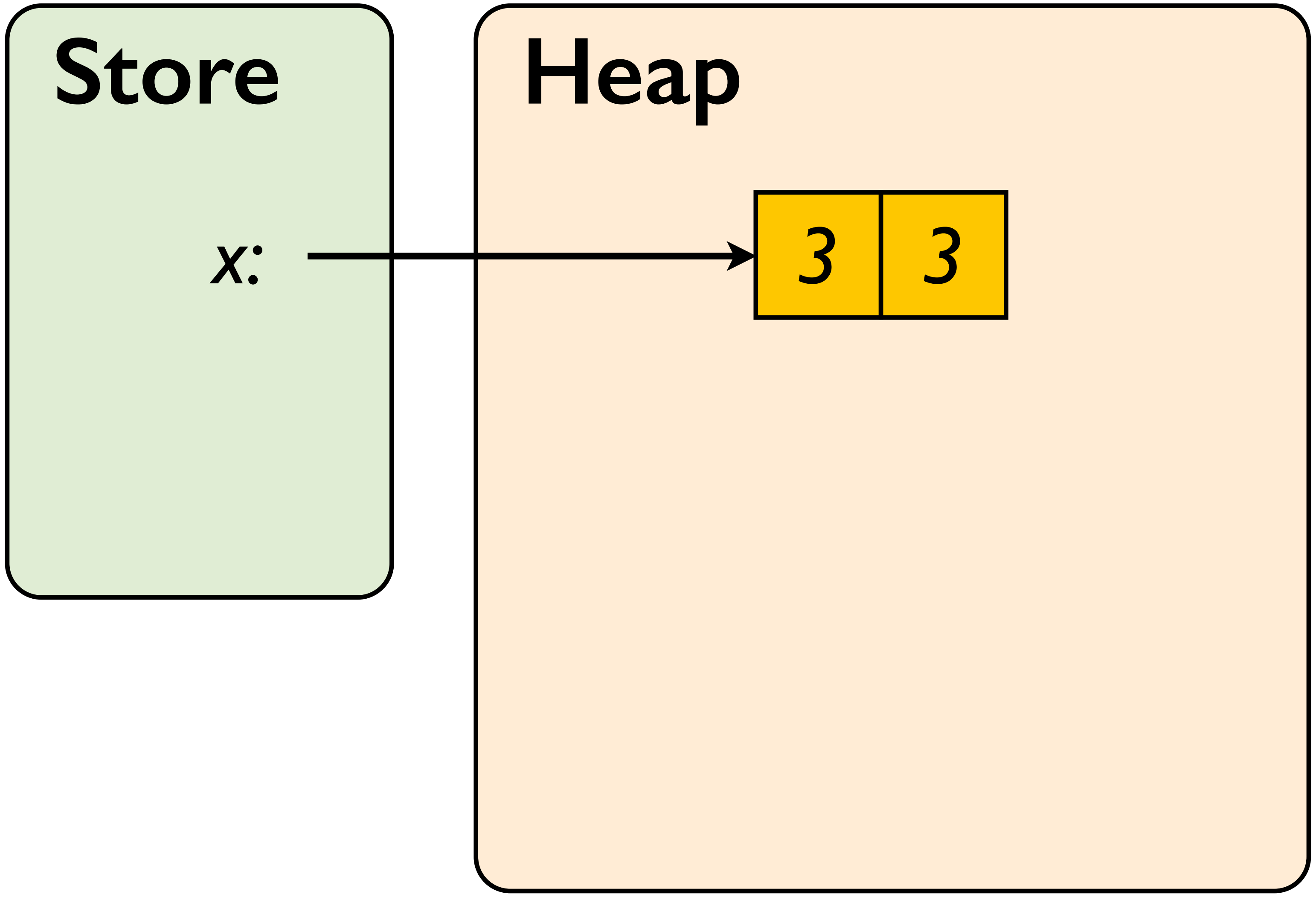
{y|->4 * true}



Proof outline

{emp}

x := cons(3,3);

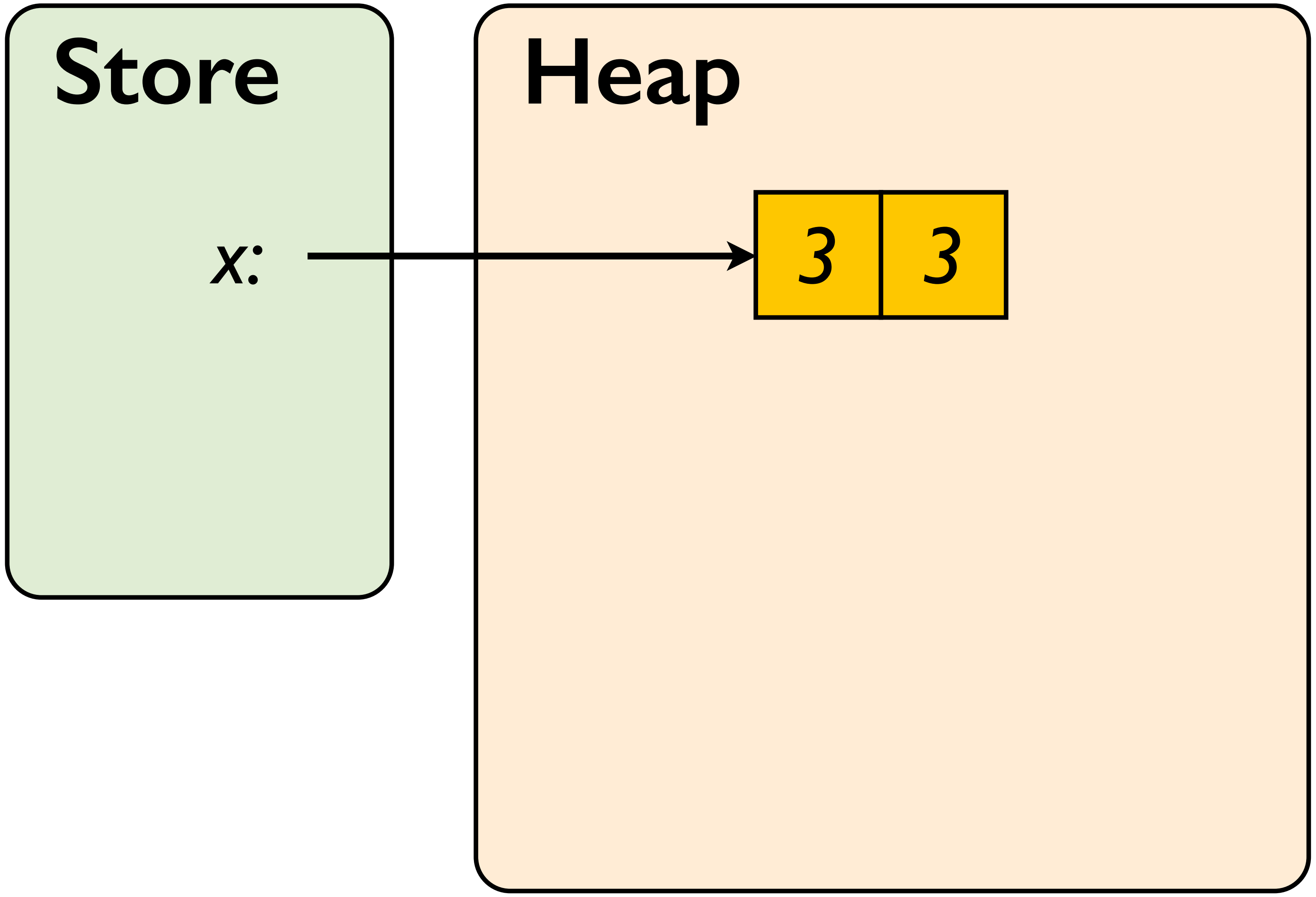


Proof outline

{emp}

x := cons(3,3);

{x |-> 3,3}



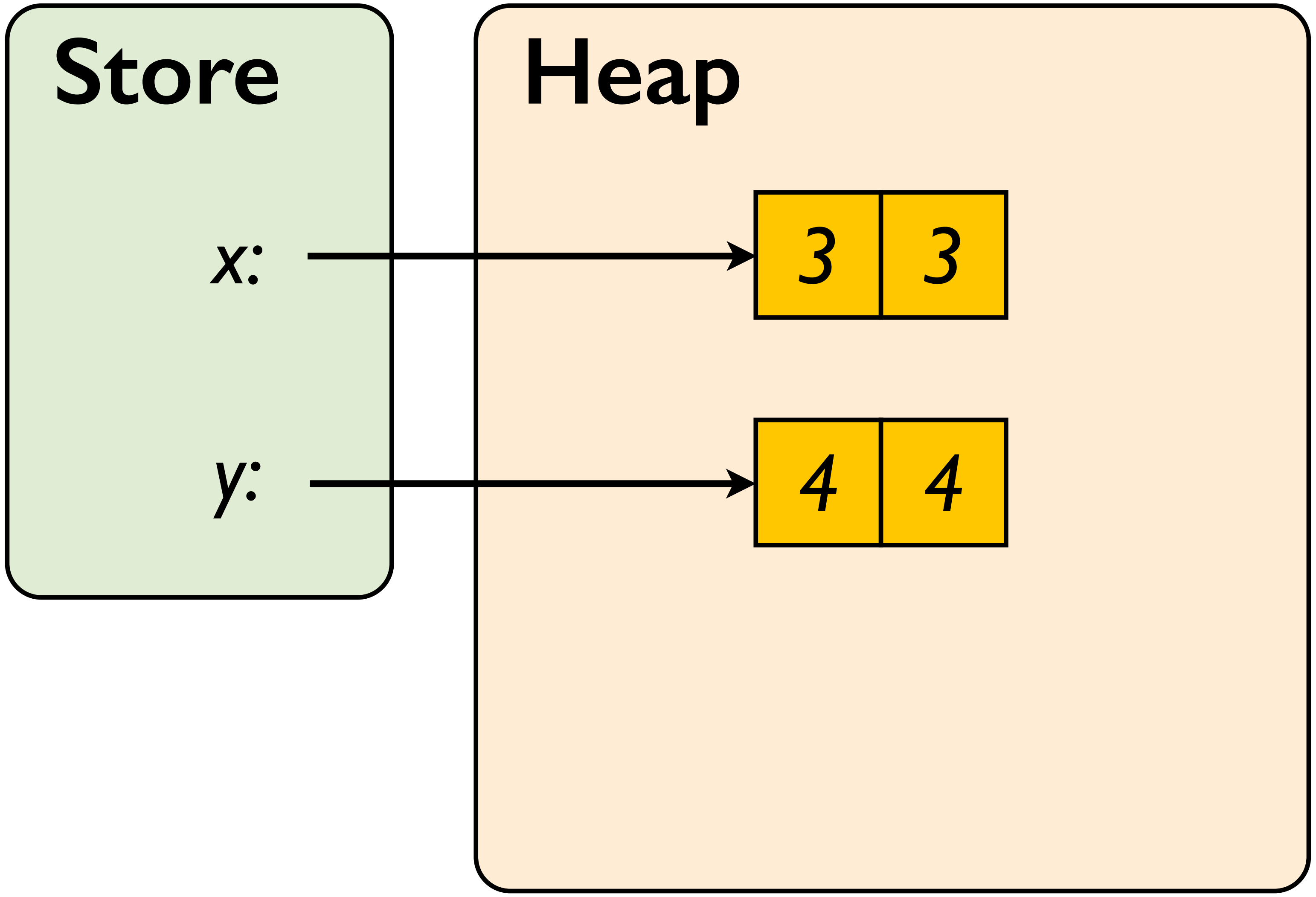
Proof outline

{emp}

x := cons(3,3);

{x |-> 3,3}

y := cons(4,4);



Proof outline

{emp}

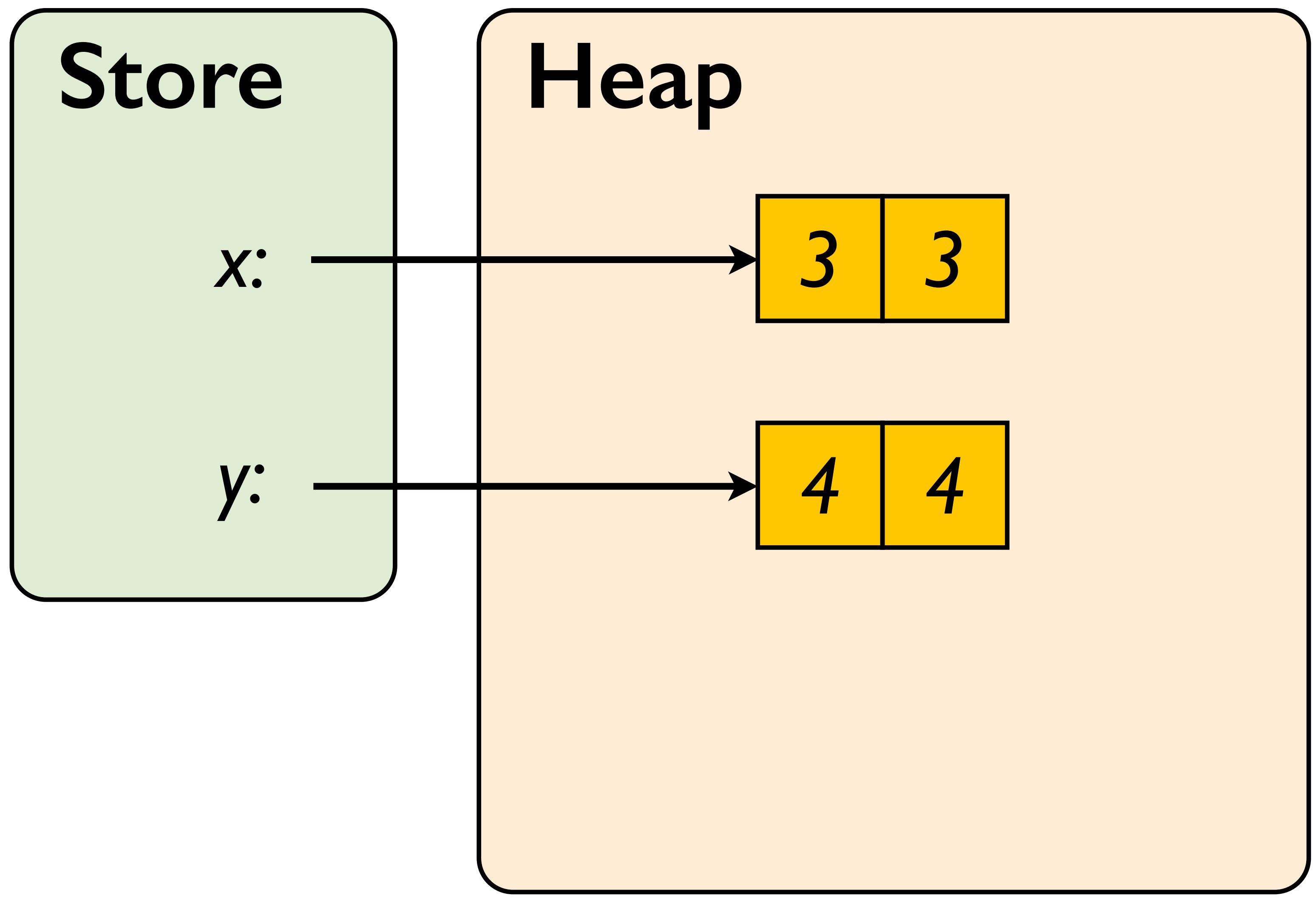
    x := cons(3,3);

{x |-> 3,3}

{x |-> 3,3 * emp}

    y := cons(4,4);

rule of consequence



Proof outline

{emp}

    x := cons(3,3);

{x |-> 3,3}

{x |-> 3,3 * emp}

> {emp}
>
>   y := cons(4,4);
>
> {y |-> 4,4}

frame rule!

{x |-> 3,3 * y |-> 4,4}

Proof outline

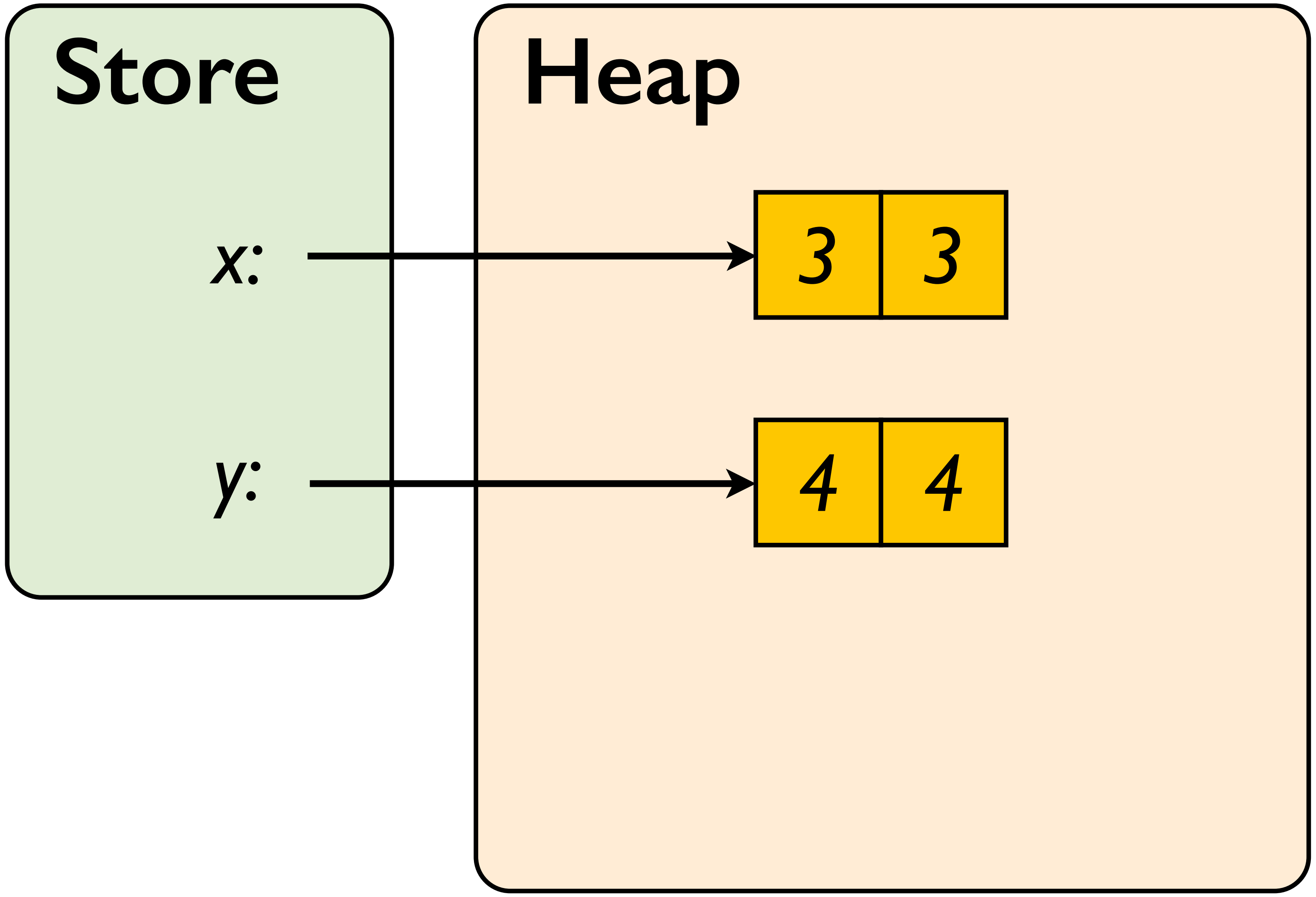
{emp}

x := cons(3,3);

{x |-> 3,3}

y := cons(4,4);

{x |-> 3,3 * y |-> 4,4}



Proof outline

{emp}

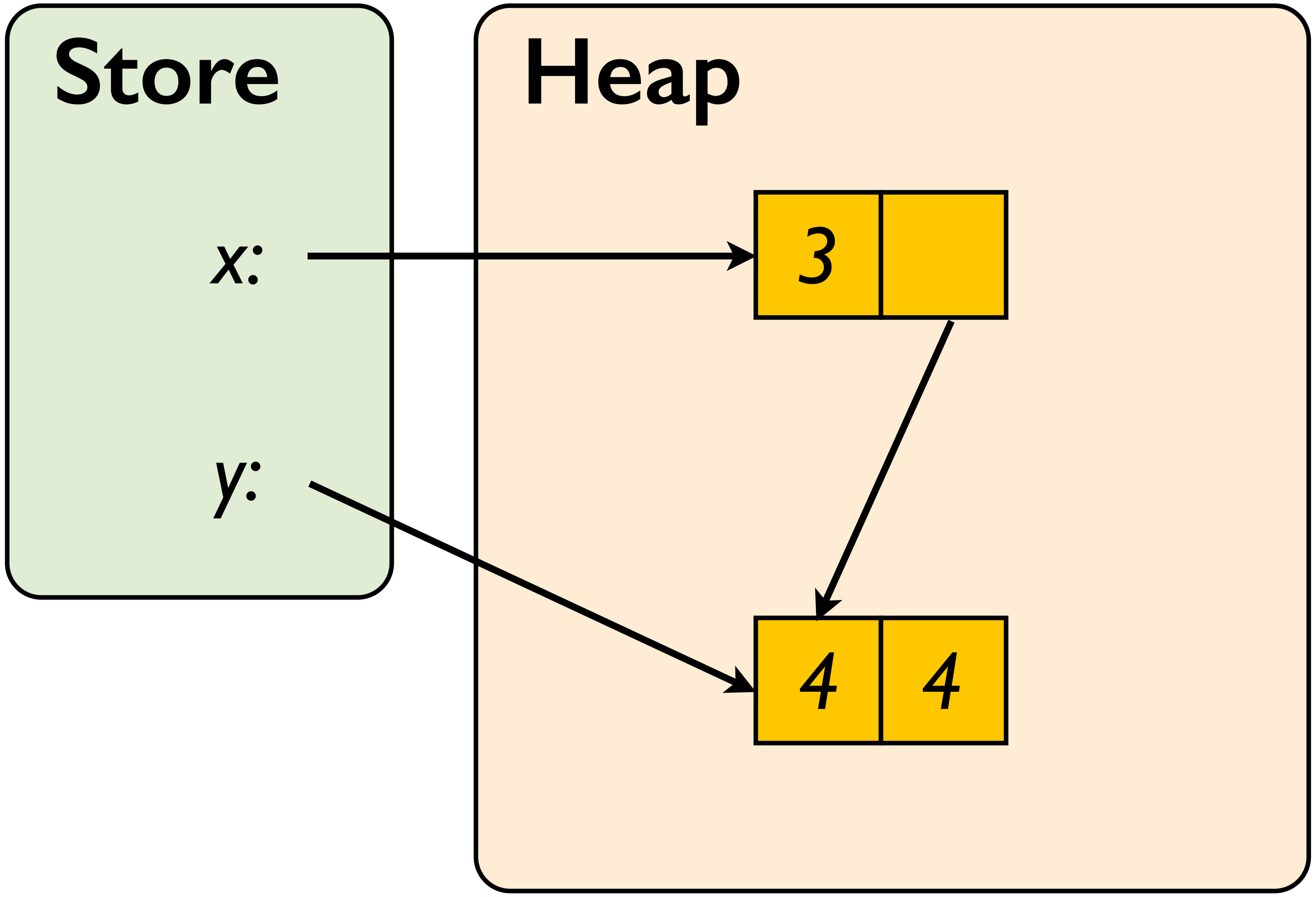
  x := cons(3,3);

{x |-> 3,3}

  y := cons(4,4);

{x |-> 3,3 * y |-> 4,4}

  [x+1] := y;



Proof outline

{emp}

x := cons(3,3);

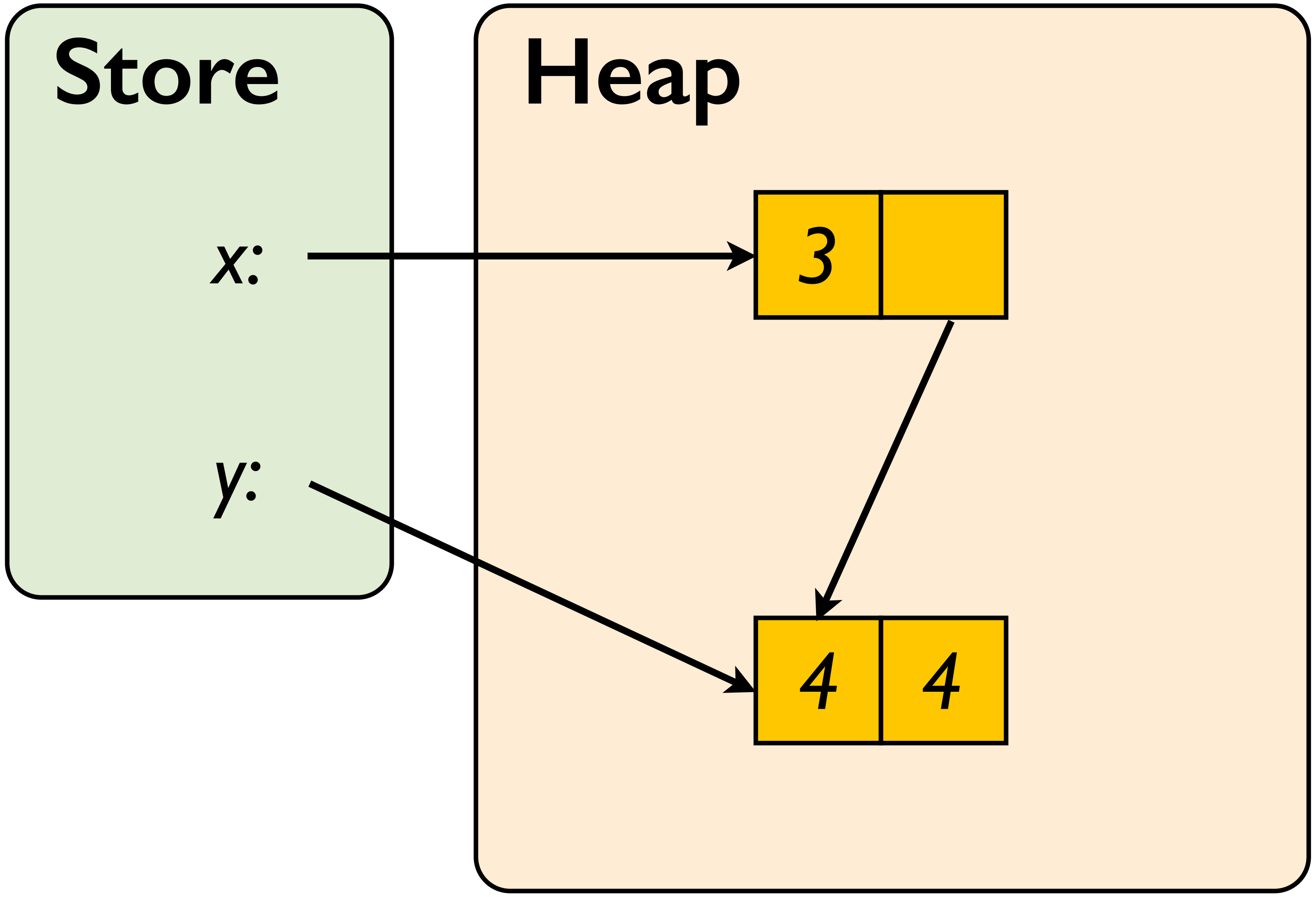
{x |-> 3,3}

y := cons(4,4);

{x |-> 3,3 * y |-> 4,4}

{x |-> 3 * x+1 |-> 3
   * y |-> 4,4}

[x+1] := y;



Proof outline

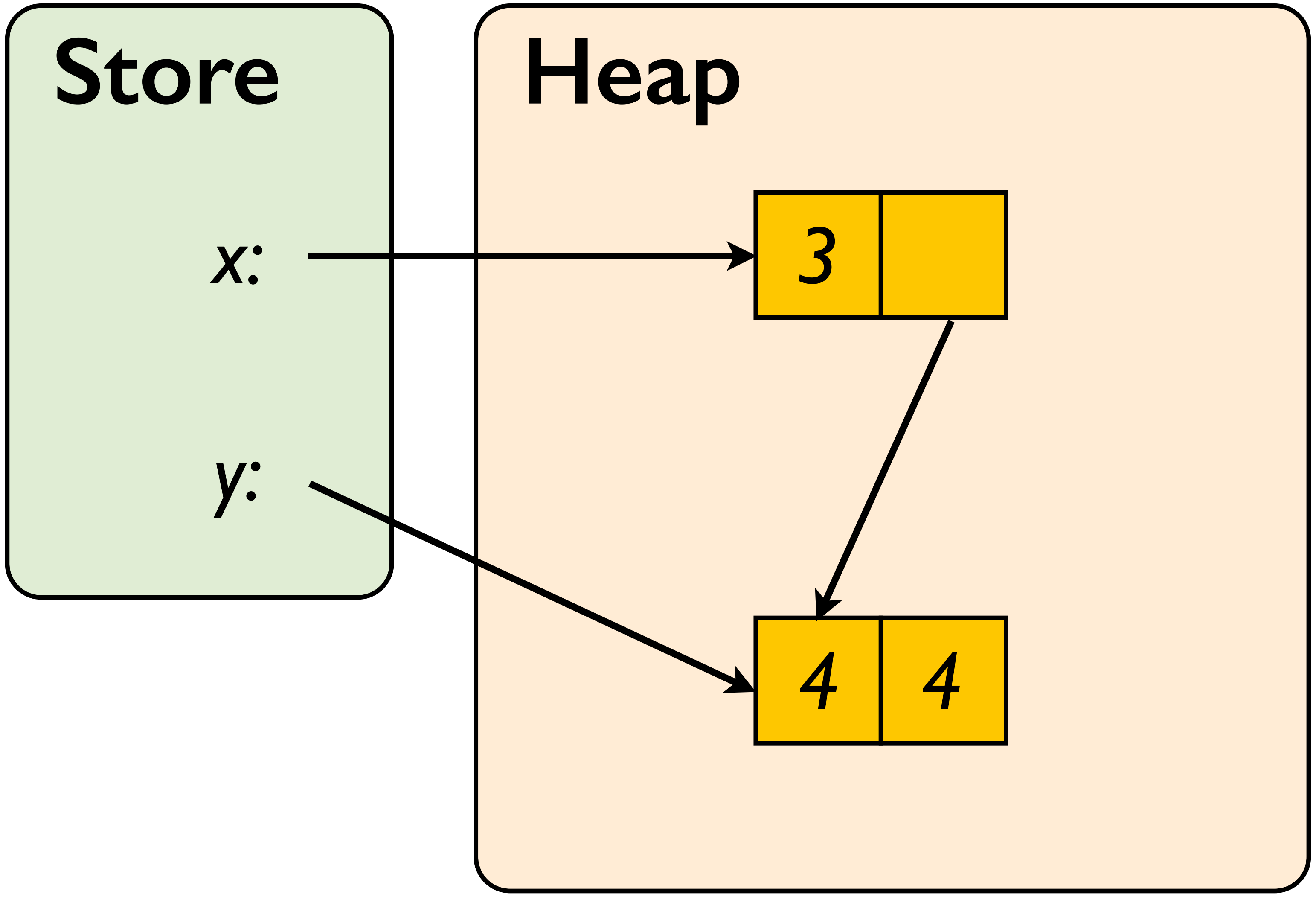
{emp}
  x := cons(3,3);
{x |-> 3,3}
  y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
{x |-> 3 * x+1 |-> 3
  * y |-> 4,4}
  [x+1] := y;
{x |-> 3 * x+1 |-> y
  * y |-> 4,4}

{x+1 |-> 3}
  [x+1] := y;
{x+1 |-> y}

frame rule!

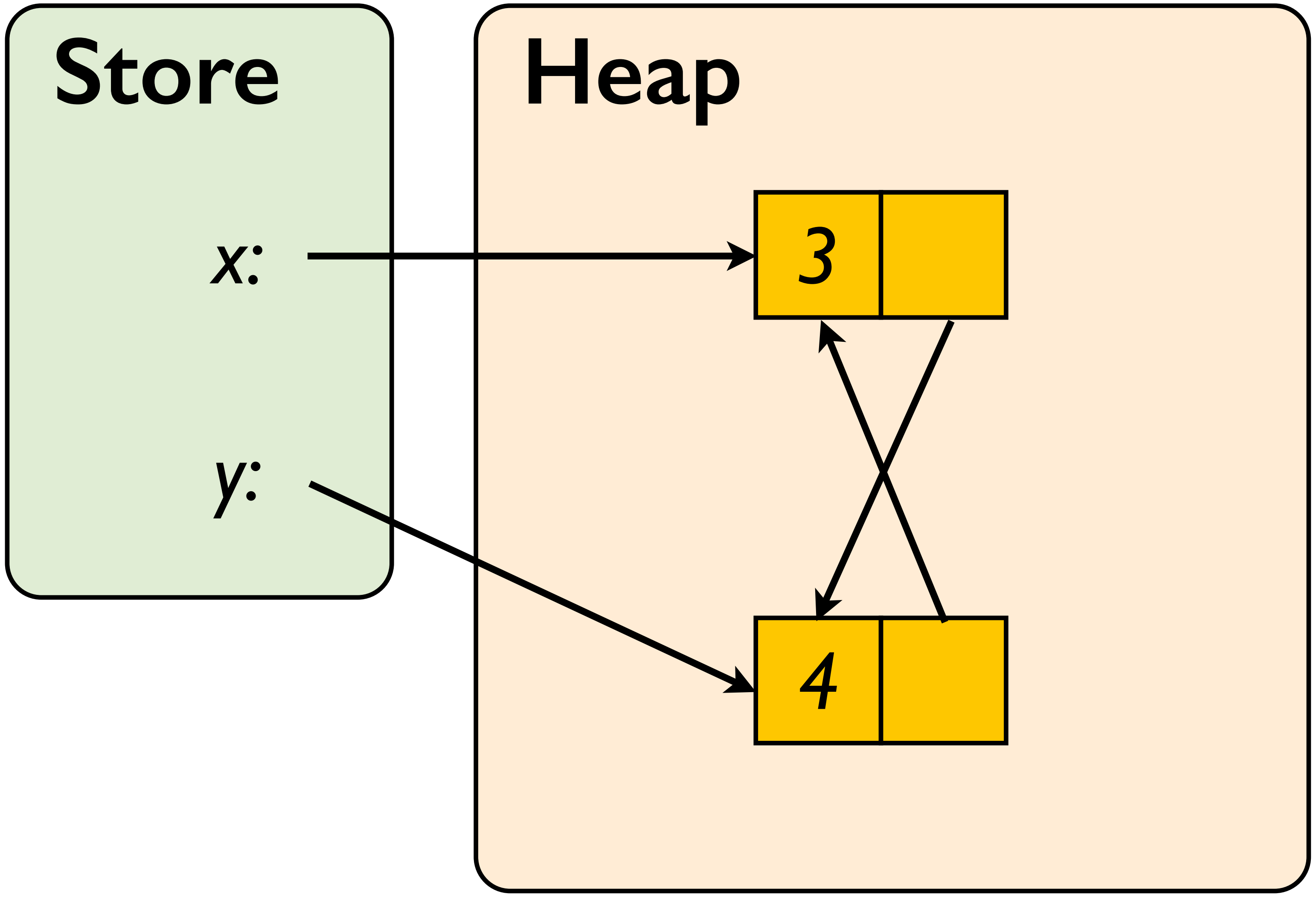
Proof outline

{emp}
    x := cons(3,3);
{x |-> 3,3}
    y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
    [x+1] := y;
{x |-> 3,y * y |-> 4,4}



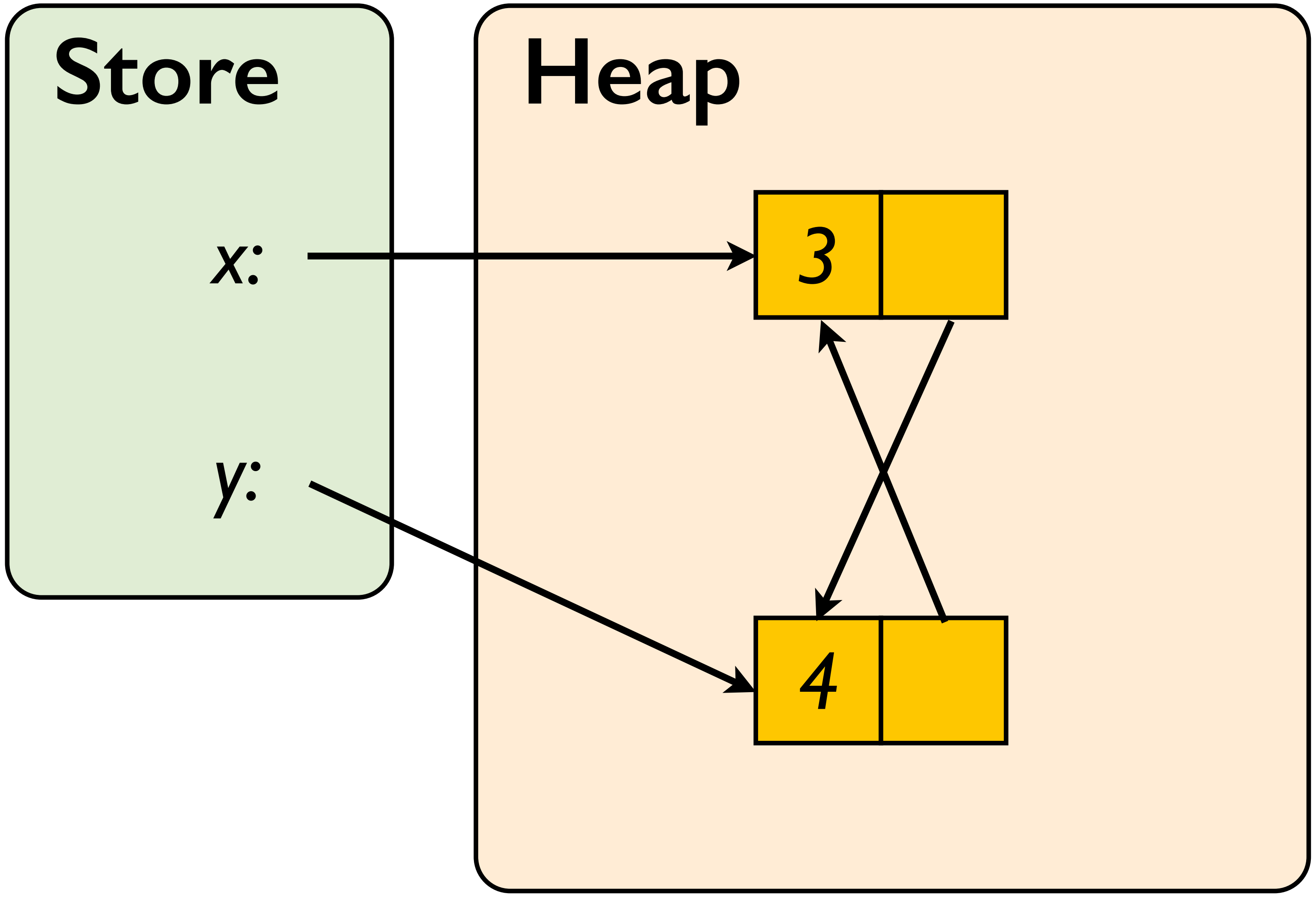
Proof outline

{emp}
x := cons(3,3);
{x |-> 3,3}
y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
[x+1] := y;
{x |-> 3,y * y |-> 4,4}
[y+1] := x;



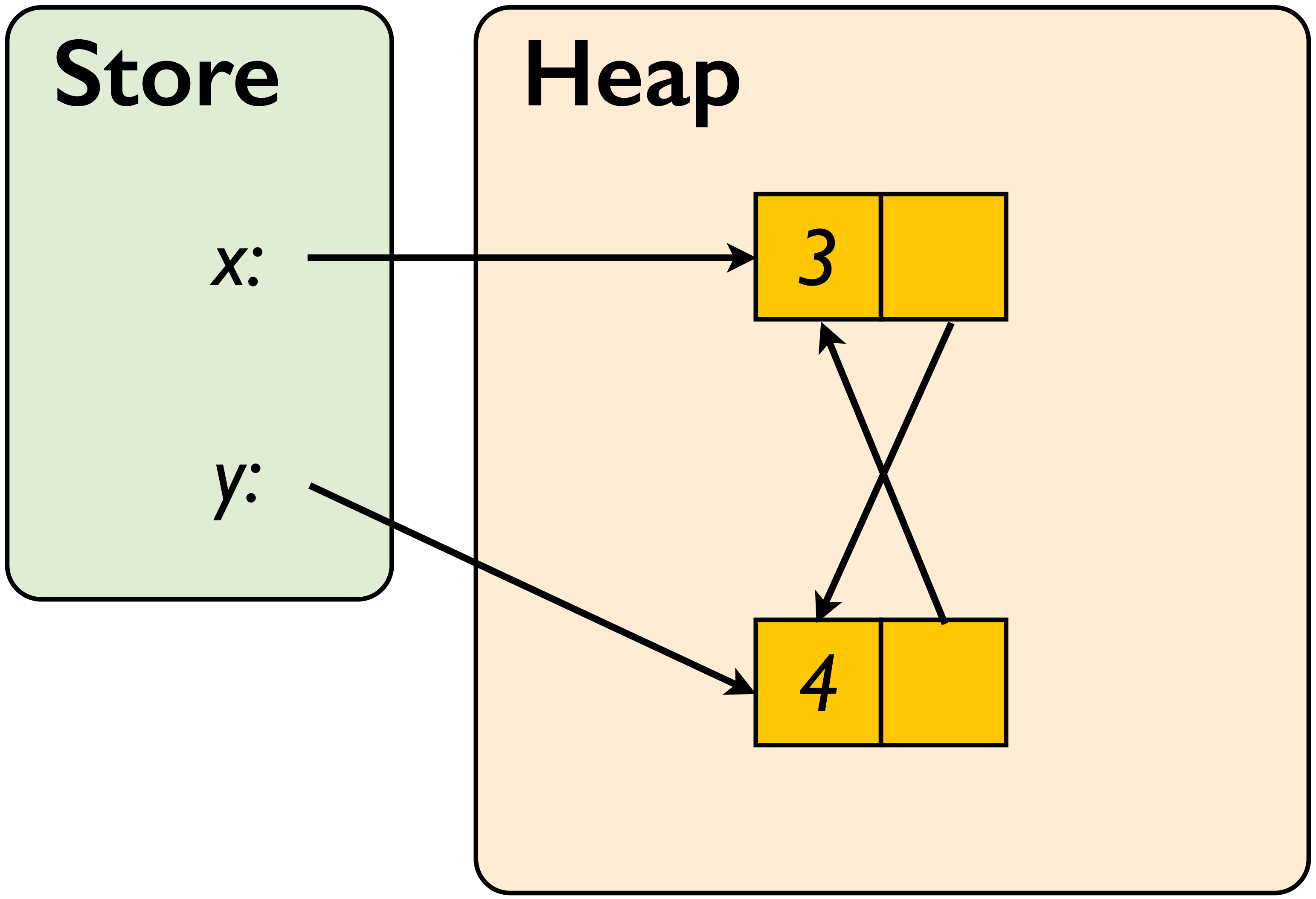
Proof outline

{emp}
  x := cons(3,3);
{x |-> 3,3}
  y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
  [x+1] := y;
{x |-> 3,y * y |-> 4,4}
  [y+1] := x;
{x |-> 3,y * y |-> 4,x}



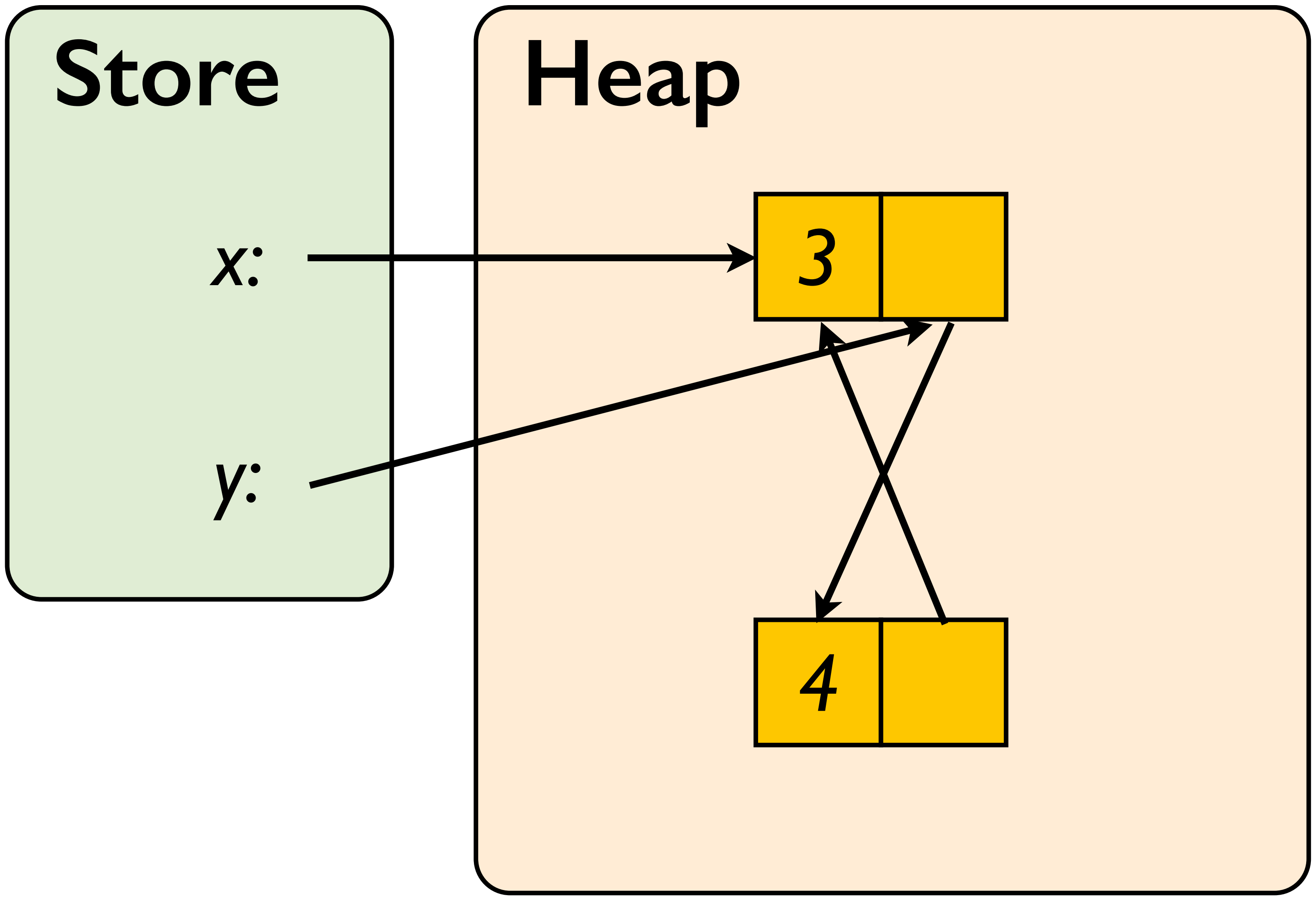
{emp}
x := cons(3,3);
{x |-> 3,3}
y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
[x+1] := y;
{x |-> 3,y * y |-> 4,4}
[y+1] := x;
{x |-> 3,y * y |-> 4,x}

Proof outline



Proof outline

```
{emp}
   x := cons(3,3);
{x |-> 3,3}
   y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
   [x+1] := y;
{x |-> 3,y * y |-> 4,4}
   [y+1] := x;
{x |-> 3,y * y |-> 4,x}
   y := x+1;
```



Proof outline

```
{emp}
   x := cons(3,3);
{x |-> 3,3}
   y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
   [x+1] := y;
{x |-> 3,y * y |-> 4,4}
   [y+1] := x;
{x |-> 3,y * y |-> 4,x}
   y := x+1;
```

via "forward" assignment axiom from Hoare logic (y^{old} is implicitly ∃-quantified)

{x |-> 3,y * y |-> 4,x}

y := x+1

{x |-> 3,y^{old} * y^{old} |-> 4,x ∧ y = x+1}

{emp}

x := cons(3,3);

{x |-> 3,3}

y := cons(4,4);

{x |-> 3,3 * y |-> 4,4}

[x+1] := y;

{x |-> 3,y * y |-> 4,4}

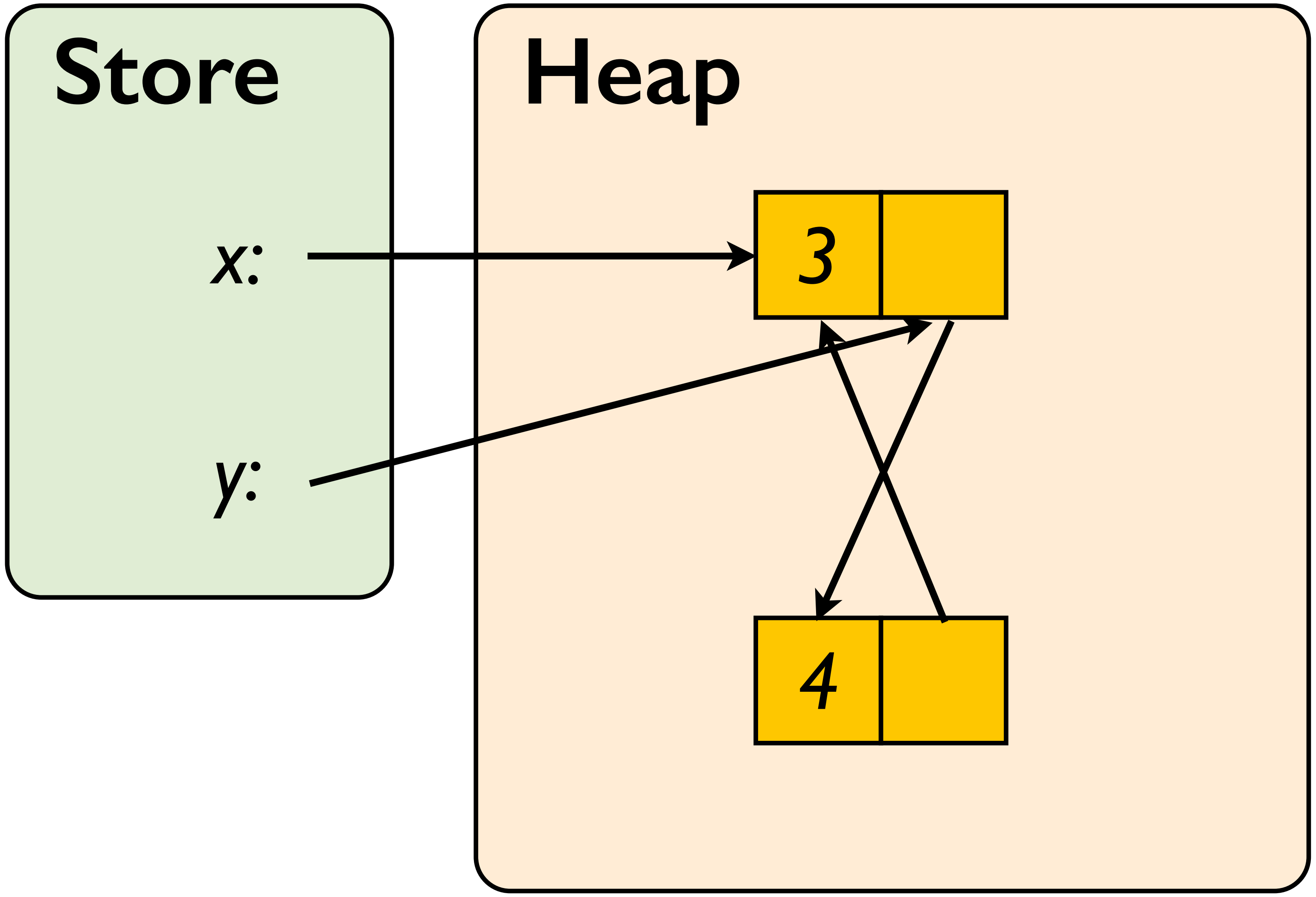
[y+1] := x;

{x |-> 3,y * y |-> 4,x}

y := x+1;

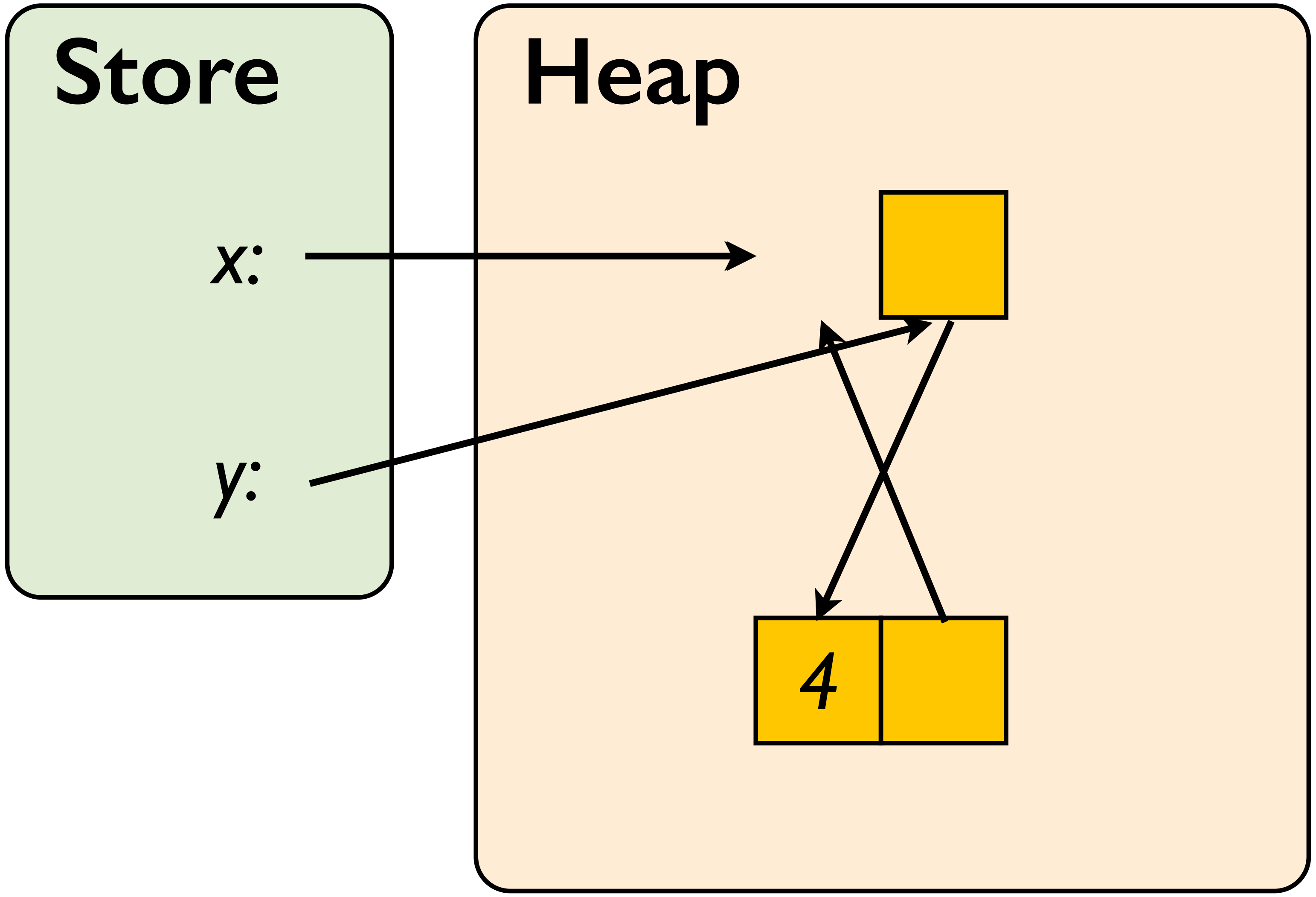
{x |-> 3,y^{old} * y^{old} |-> 4,x ∧ y = x+1}

Proof outline



{emp}
x := cons(3,3);
{x |-> 3,3}
y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
[x+1] := y;
{x |-> 3,y * y |-> 4,4}
[y+1] := x;
{x |-> 3,y * y |-> 4,x}
y := x+1;
{x |-> 3,y^{old} * y^{old} |-> 4,x
∧ y = x+1}
dispose x;

Proof outline



Proof outline

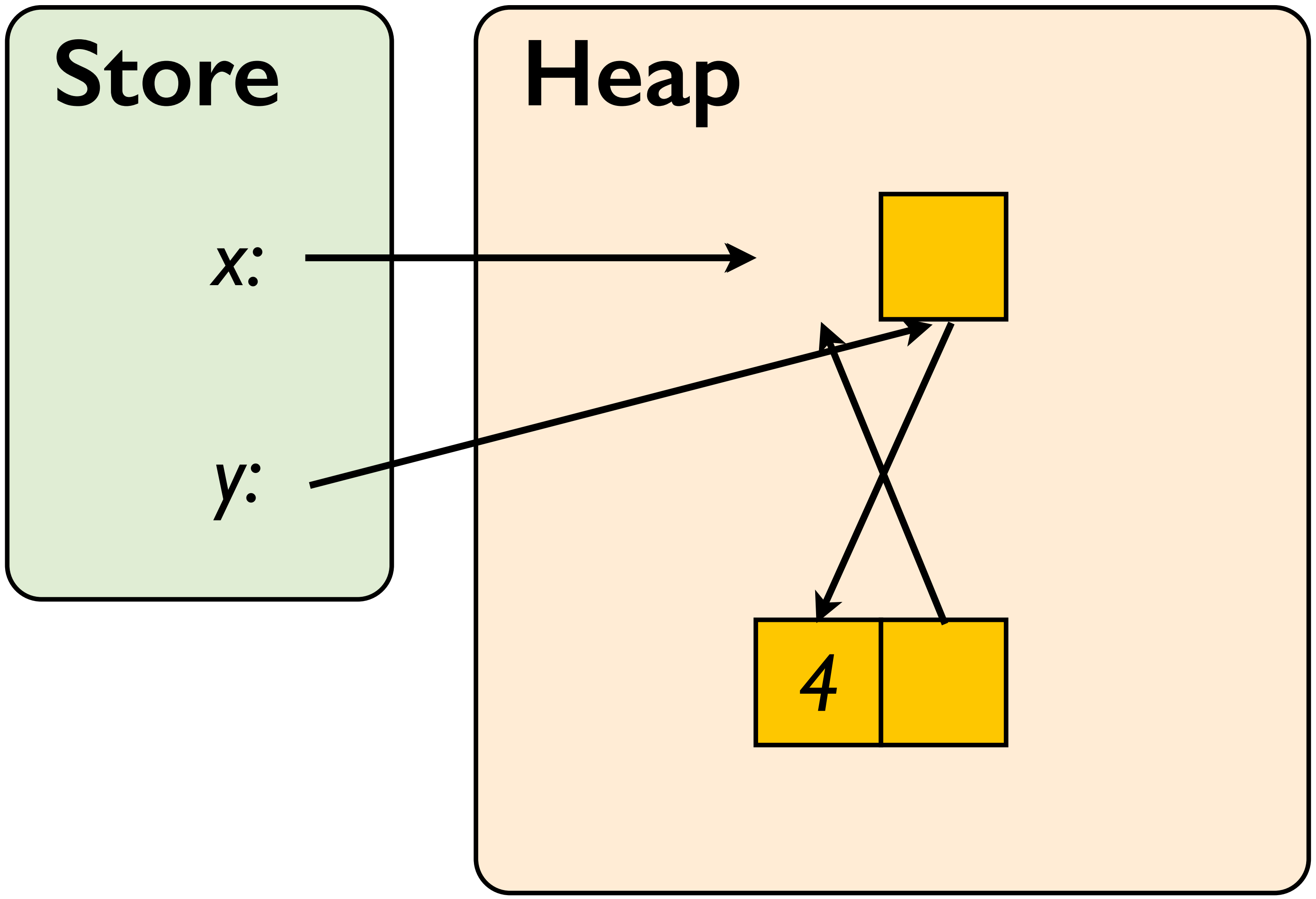
{emp}
  x := cons(3,3);
{x |-> 3,3}
  y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
  [x+1] := y;
{x |-> 3,y * y |-> 4,4}
  [y+1] := x;
{x |-> 3,y * y |-> 4,x}
  y := x+1;
{x |-> 3,y^{old} * y^{old} |-> 4,x
  ∧ y = x+1}
  dispose x;
{emp * x+1 |-> y^{old} *
y^{old} |-> 4,x ∧ y = x+1}

{x |-> 3}
  dispose x;
{emp}

frame rule!

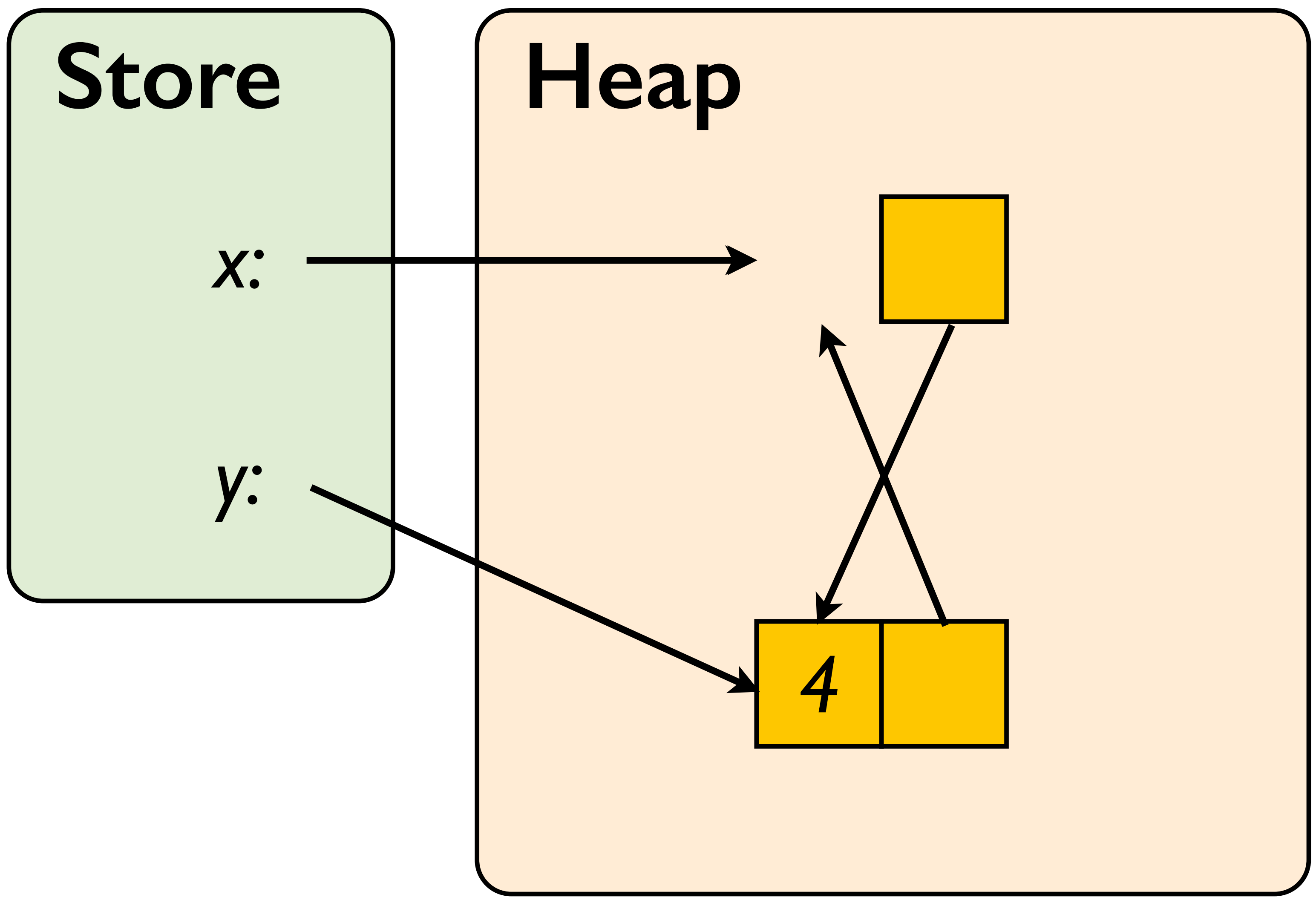
Proof outline

{emp}
   x := cons(3,3);
{x |-> 3,3}
   y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
   [x+1] := y;
{x |-> 3,y * y |-> 4,4}
   [y+1] := x;
{x |-> 3,y * y |-> 4,x}
   y := x+1;
{x |-> 3,y^{old} * y^{old} |-> 4,x
   ∧ y = x+1}
   dispose x;
{x+1 |-> y^{old} * y^{old} |-> 4,x
   ∧ y = x+1}



Proof outline

{emp}
x := cons(3,3);
{x |-> 3,3}
y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
[x+1] := y;
{x |-> 3,y * y |-> 4,4}
[y+1] := x;
{x |-> 3,y * y |-> 4,x}
y := x+1;
{x |-> 3,y^{old} * y^{old} |-> 4,x
∧ y = x+1}
dispose x;
{x+1 |-> y^{old} * y^{old} |-> 4,x
∧ y = x+1}
y := [y];



Proof outline

{emp}
x := cons(3,3);
{x |-> 3,3}
y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
[x+1] := y;
{x |-> 3,y * y |-> 4,4}
[y+1] := x;
{x |-> 3,y * y |-> 4,x}
y := x+1;
{x |-> 3,y^{old} * y^{old} |-> 4,x
∧ y = x+1}
dispose x;
{x+1 |-> y^{old} * y^{old} |-> 4,x
∧ y = x+1}
y := [y];

frame rule and consequence!

{x+1 = y ∧ y |-> y^{old} }
y := [y];
{x+1 |-> y^{old} ∧ y^{old} = y}

Proof outline

{emp}

x := cons(3,3);

{x |-> 3,3}

y := cons(4,4);

{x |-> 3,3 * y |-> 4,4}

[x+1] := y;

{x |-> 3,y * y |-> 4,4}

[y+1] := x;

{x |-> 3,y * y |-> 4,x}

y := x+1;

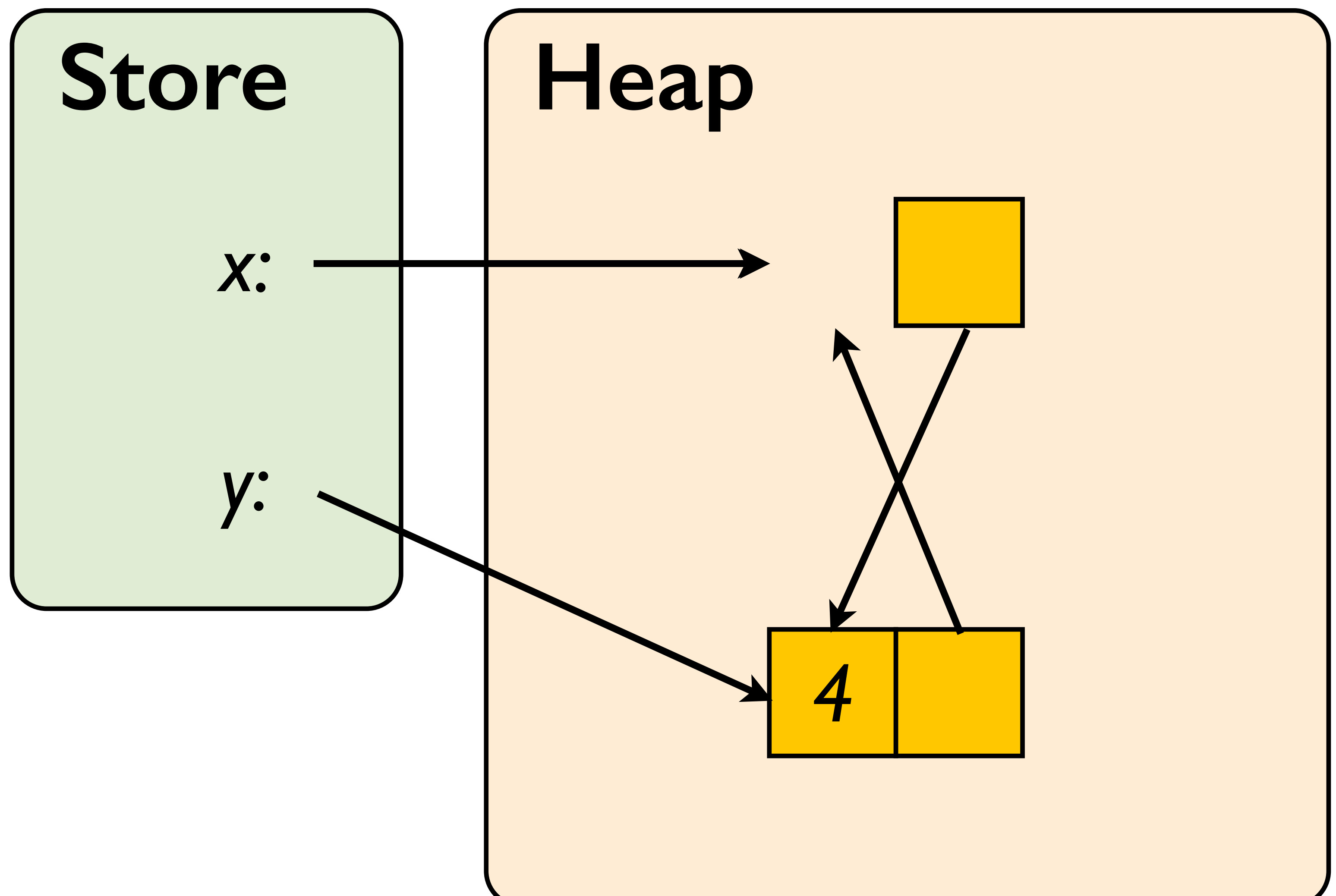
{x |-> 3,y^{old} * y^{old} |-> 4,x
∧ y = x+1}

dispose x;

{x+1 |-> y^{old} * y^{old} |-> 4,x
∧ y = x+1}

y := [y];

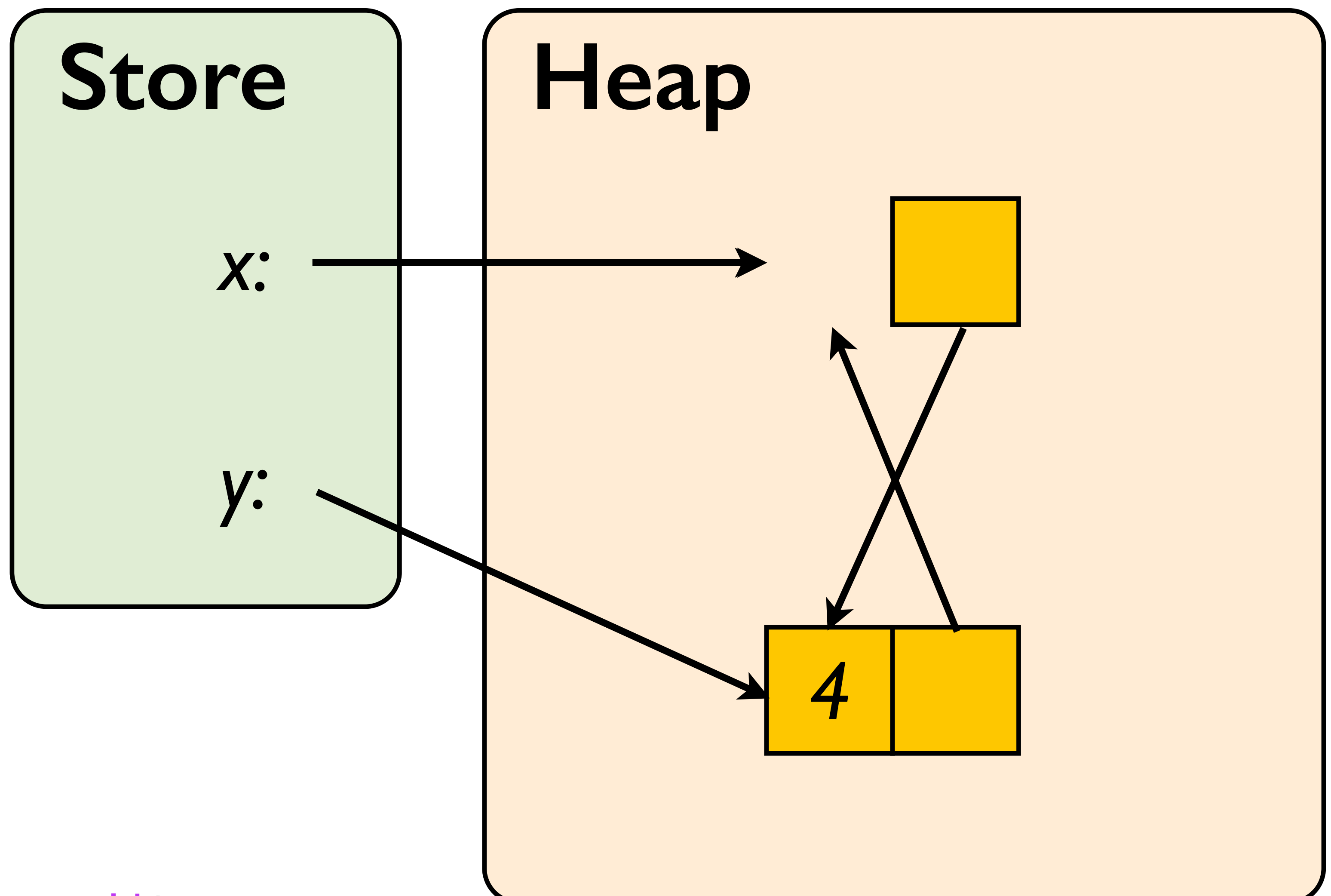
{x+1 |-> y^{old} * y^{old} |-> 4,x ∧ y = y^{old} }



{emp}
x := cons(3,3);
{x |-> 3,3}
y := cons(4,4);
{x |-> 3,3 * y |-> 4,4}
[x+1] := y;
{x |-> 3,y * y |-> 4,4}
[y+1] := x;
{x |-> 3,y * y |-> 4,x}
y := x+1;
{x |-> 3,y^{old} * y^{old} |-> 4,x
∧ y = x+1}
dispose x;
{x+1 |-> y^{old} * y^{old} |-> 4,x
∧ y = x+1}
y := [y];
{x+1 |-> y^{old} * y^{old} |-> 4,x ∧ y = y^{old}}

{y |-> 4 * true}

Proof outline

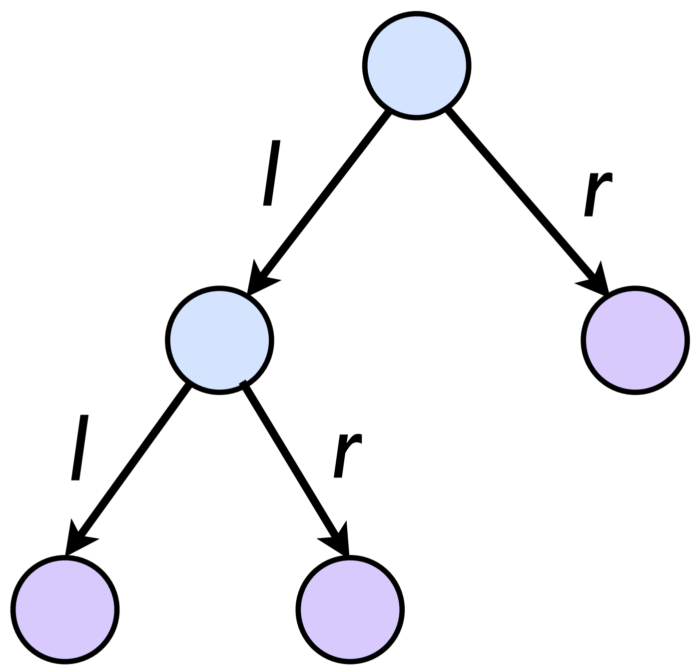


Inductive predicates in assertions

- heap portions in more realistic programs might comprise data structures such as trees, linked lists, …
- helpful and concise to reason about such structures using inductively-defined predicates

Tree disposal

```
procedure DispTree(p)
local i, j;
if ¬isatom?(p) then
   i := p→l;
   j := p→r;
   DispTree(i)
   DispTree(j)
   free(p)
```



Tree predicate

$$\text{tree}(e) \iff \texttt{if } \text{isAtom}(e) \texttt{ then } \text{emp} \;\; \texttt{else } \exists x, y.\ e \mapsto x, y * \text{tree}(x) * \text{tree}(y)$$

- notes:
  - isAtom(e) returns true if e is an atomic value (e.g. number, characters) and <u>not</u> a location
  - if-then-else is easily compilable to logic (how?)

Tree disposal

```
procedure DispTree(p)
local i, j;
if ¬isatom?(p) then
   i := p→l;
   j := p→r;
   DispTree(i)
   DispTree(j)
   free(p)
```

{tree(p)} DispTree(p) {emp}

Tree disposal

```
procedure DispTree(p)
local i, j;
if ¬isatom?(p) then
    i := p→l;
    j := p→r;
    DispTree(i)
    DispTree(j)
    free(p)
```

we first:

- *adjust for our store/heap model*
- *focus the proof on the crucial part*

$\{\text{tree}(p)\}\ \text{DispTree}(p)\ \{\text{emp}\}$

Tree disposal proof

(from O'Hearn)

```
{p |-> x,y * tree(x) * tree(y)}
    i := [p];
    j := [p+1];
    DispTree(i);
    DispTree(j);
    dispose(p);
    dispose(p+1);
{emp}
```

Tree disposal proof

(from O'Hearn)

{p |-> x,y * tree(x) * tree(y)}

i := [p];

Tree disposal proof

(from O'Hearn)

{p |-> x,y * tree(x) * tree(y)}

i := [p];

{p |-> x}

i := [p]

{p |-> x ∧ x = i }

frame rule!

{p |-> x,y * tree(x) * tree(y) ∧ x = i}

{p |-> x,y * tree(i) * tree(y)}

Tree disposal proof

(from O'Hearn)

{p |-> x,y * tree(x) * tree(y)}

    i := [p];

{p |-> x,y * tree(i) * tree(y)}

    j := [p+1];

Tree disposal proof

(from O'Hearn)

{p |-> x,y * tree(x) * tree(y)}

i := [p];

{p |-> x,y * tree(i) * tree(y)}

j := [p+1];

{p |-> x,y * tree(i) * tree(j)}

Tree disposal proof

(from O'Hearn)

```
{p |-> x,y * tree(x) * tree(y)}
    i := [p];
{p |-> x,y * tree(i) * tree(y)}
    j := [p+1];
{p |-> x,y * tree(i) * tree(j)}
    DispTree(i);
```

Tree disposal proof

(from O'Hearn)

{p |-> x,y * tree(x) * tree(y)}

    i := [p];

{p |-> x,y * tree(i) * tree(y)}

    j := [p+1];

{p |-> x,y * tree(i) * tree(j)}

    DispTree(i);

{p |-> x,y * emp * tree(j)}

frame rule!

{tree(i)}

    DispTree(i)

{emp}

Tree disposal proof

(from O'Hearn)

```
{p |-> x,y * tree(x) * tree(y)}
   i := [p];
{p |-> x,y * tree(i) * tree(y)}
   j := [p+1];
{p |-> x,y * tree(i) * tree(j)}
   DispTree(i);
{p |-> x,y * emp * tree(j)}
   DispTree(j);
```

Tree disposal proof
(from O'Hearn)

```
{p |-> x,y * tree(x) * tree(y)}
    i := [p];
{p |-> x,y * tree(i) * tree(y)}
    j := [p+1];
{p |-> x,y * tree(i) * tree(j)}
    DispTree(i);
{p |-> x,y * emp * tree(j)}
    DispTree(j);
{p |-> x,y * emp * emp}
```

```
{p |-> x,y * tree(x) * tree(y)}
    i := [p];
{p |-> x,y * tree(i) * tree(y)}
    j := [p+1];
{p |-> x,y * tree(i) * tree(j)}
    DispTree(i);
{p |-> x,y * emp * tree(j)}
    DispTree(j);
{p |-> x,y * emp * emp}
```

```
{p |-> x,y * tree(x) * tree(y)}
    i := [p];
{p |-> x,y * tree(i) * tree(y)}
    j := [p+1];
{p |-> x,y * tree(i) * tree(j)}
    DispTree(i);
{p |-> x,y * emp * tree(j)}
    DispTree(j);
{p |-> x,y * emp * emp}
    dispose(p);
    dispose(p+1);
```

```
{p |-> x,y * tree(x) * tree(y)}
    i := [p];
{p |-> x,y * tree(i) * tree(y)}
    j := [p+1];
{p |-> x,y * tree(i) * tree(j)}
    DispTree(i);
{p |-> x,y * emp * tree(j)}
    DispTree(j);
{p |-> x,y * emp * emp}
    dispose(p);
    dispose(p+1);
{emp * emp * emp}
```

```
{p |-> x,y * tree(x) * tree(y)}
    i := [p];
{p |-> x,y * tree(i) * tree(y)}
    j := [p+1];
{p |-> x,y * tree(i) * tree(j)}
    DispTree(i);
{p |-> x,y * emp * tree(j)}
    DispTree(j);
{p |-> x,y * emp * emp}
    dispose(p);
    dispose(p+1);
{emp * emp * emp}
{emp}
```

Next on the agenda

(1) model of program states for separation logic ✓

(2) assertions and spatial connectives ✓

(3) axioms and inference rules ✓

(4) program proofs ✓

In a nutshell 

The frame rule is absolutely key to separation logic proofs

$$\frac{\{p\} \quad C \quad \{q\}}{\{p * r\} \quad C \quad \{q * r\}}$$

Thank you! Questions?